Boundary Control of an Axially Moving String : Actuator Dynamics Included

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In this paper, an active vibration control of a translating tensioned string with the use of an electro-hydraulic servo mechanism at the right boundary is investigated. The equations of motion of the string are derived by using Hamilton's principle for the systems with changing mass. The control objective is to suppress the transverse vibrations of the string via a right-boundary control. An energy-based right-boundary control law, generating a specific current input to the servo-valve, is derived. It is revealed that a time-varying boundary force, as a function of the slope of the string at the right end and a suitably chosen damping coefficient of the actuator, can successfully suppress the transverse vibrations. The exponential stability of the closed loop system is proved. The effectiveness of the proposed control law is demonstrated via simulations.

Key Words : Axially Moving System, Exponential Stability, Boundary Control, Hyperbolic Partial Differential Equation, Lyapunov Method

1. Introduction

The control problems of axially moving systems occur in various engineering areas : For example, the strips in thin metal-sheet production lines, the cables, belts, and chains in power transmission lines, the magnetic tapes in recorders, the band saws, etc. The dynamics of these systems can be differently modeled depending on the length, flexibility, and control objectives of the system considered. For instance, the dynamics of a moving cable of an elevator can be described by a string equation, but that of a rubber belt in the traditional mill can be well represented by a belt equation. The difference between a string and a belt lies in whether the longitudinal elongation is considered or not.

In axially moving systems, the transverse (lateral) vibration of the moving material often causes a serious problem in achieving good quality. It is also known that these vibrations are often caused by the eccentricity of a pulley, and/or an irregular speed of the driving motor, and/or a non-uniform material property, and/or environmental disturbances. Since the quality requirement as well as the productivity in a production line is getting stricter, an active or a semi-active vibration control is nowadays se-

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riously considered.

Diverse results on the dynamics, stability, and/or active/passive controls for axially moving systems have appeared in the literature (Carrier, 1945 ; Bapat and Srinivasan, 1967 ; Wickert and Mote, 1990 ; Wickert, 1992 ; Oshima et al., 1997 ; Pellicano and Zirilli, 1998 ; Shahruz, 1998 ; 2000 ; Osstveen and Curtain, 2000 ; Kim and Yoo, 2002 ; Lee, 2002 ; Sohn et al., 2002 ; Kim et al., 2002 ; Matsuno et al., 2002 ; Qu, 2002 ; Choi et al., 2004 ; Hong et al., 2003 ; 2004 ; Yang et al., 2004a, b). Particularly, Mote (1965) modeled the dynamics of a band saw, as an axially moving string, and investigated its instability in relation to the moving speed and excitation frequency of the saw. Wickert and Mote (1988) reported a passive control strategy, by changing its damping and stiffness, for axially moving continua. Morgul (1992) investigated a boundary control law that suppresses the lateral vibration of an Euler-Bernoulli beam, but in his work the beam itself was not axially moving. Laousy et al. (1996) investigated a boundary feedback stabilization method for a rotating body-beam system. Lee and Mote (1996) derived an optimal boundary force control law that dissipates the vibration energy of an axially moving string. Fung et al. (1999a, b) reported boundary control laws for linear and nonlinear strings, in which the dynamics of the actuator has been incorporated in the control law design. An optimal control and an adaptive control of an axially moving string were investigated in (Fung et al., 2002a, b), respectively. For a translating linear beam, Lee and Mote (1999) analyzed the wave characteristics of the beam and derived optimal boundary damping laws as a function of linear velocity, linear slope, and linear force. Li and Rahn (2000) investigated an adaptive vibration control for an axially moving linear beam by splitting the moving part into two spans, a controlled span and an uncontrolled span. Li et al. (2002) applied the control strategy of Li and Rahn (2000) to a linear string, providing experimental results. Fard and Sagatun (2001) focused on the exponential stabilization of a nonlinear beam, not axially moving, by a boundary control.

The contributions of this paper are the following. First, the actuator dynamics of an axially moving string have been incorporated in the control law design. The derived control law gives a current input to the hydraulic actuator. Second, the derived boundary control law requires the feedback of two physical quantities : the string slope at the right boundary and the damping coefficient of the actuator. The string slope is measured, but the damping coefficient is a design parameter in general. Hence, once the damping coefficient is determined in the actuator design stage, the control law depends only on the slope measurement. Therefore, the use of a slope sensor enables the implementation of the control law. Finally, the exponential stability of the closed loop system has been established.

The paper is structured as follows : In Section 2, the linear string equations of motion are derived by using Hamilton's principle for the systems of changing mass. The electro-hydraulic actuator dynamics are also given. In Section 3, a stabilizing boundary control law that suppresses the transverse vibrations of the string is derived. The exponential stability of the closed loop system is proved in Section 4. In Section 5, the implementation issues of the derived control law and simulation results are discussed. Finally, Section 6 concludes the paper.

2. Equations of Motion

Figure 1 shows a schematic of the plant for analyzing dynamics and for deriving a boundary control law. The string is assumed to travel at a constant speed. The left boundary is fixed, that is, the boundary itself does not have any vertical (transversal) movement, but it allows the string to move longitudinally. However, the right boundary permits a transversal movement of the string under a control force.

Let t be the time, x be the spatial coordinate along the longitude of motion, v be the axial speed of the string, $w(x, t)$ be the transversal displacement of the string at spatial coordinate x and time t, and L be the length of the string. Then, the absolute velocity of the string at spatial

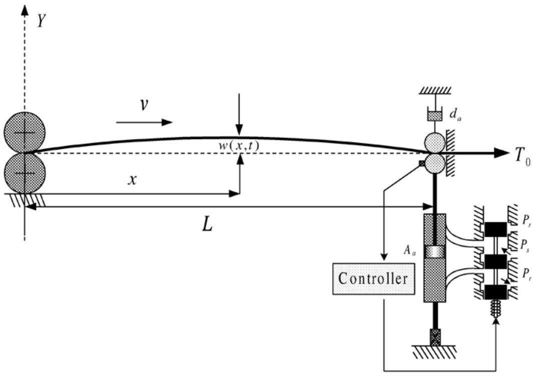


Fig. 1 An axially moving string under the right boundary control force

coordinate x is given by

$$\bar{v}=v\mathrm{i}+\frac{dw(x,\ t)}{dt}\mathrm{j} \\ =v\mathrm{i}+\{w_t(x,\ t)+vw_x(x,\ t)\}\mathrm{j} \tag{1}$$

where $(\cdot)_t=\partial(\cdot)/\partial t$ and $(\cdot)_x=\partial(\cdot)/\partial x$ denote the partial derivatives in time t and spatial coordinate x, respectively. Now, to derive the equations of motion, Hamilton's principle for the systems of changing mass (McIver, 1973) is utilized as follows:

$$\delta\int_{t_1}^{t_2}(T-U+W_{n.c.}+W_{r.b.})\,dt=0 \tag{2}$$

where T is the kinetic energy, U is the strain energy, $W_{n.c}$ is the non-conservative work, $W_{r.b}$ is the virtual momentum transport at the right boundary (no variations at the left boundary). The kinetic energy is

$$T=\frac{\rho A}{2}\int_0^L\{v^2+(w_t+vw_x)^2\}dx \\ +\frac{1}{2}mw_t^2(L,\ t) \tag{3}$$

where ρ is the mass per unit volume (material density), A is the cross-sectional area, m is the mass of the actuator, that is, of the touch roll in Fig. 1. The potential energy is

$$U=\int_0^L T_0\varepsilon_x dx \tag{4}$$

where T_0 is a constant axial tension of the strip, ε_x is the strain. The energy in (4) is due to the strip tension. If the infinitesimal distance dx is replaced by an infinitesimal length ds, the strain ε_x can be approximated as $\varepsilon_x\cong w_x^2/2$ (Benaroya, 1998). Then (4) is rewritten as

$$U=\frac{T_0}{2}\int_0^L w_x^2 dx \tag{5}$$

Now, the variations of (3) and (5), respectively, are

$$\delta T=\rho A\int_0^L(w_t+vw_x)(\delta w_t+v\delta w_x)\,dx \\ +mw_t\delta w_t(L,\ t) \tag{6}$$

$$\delta U=T_0\int_0^L w_x\delta w_x dx \tag{7}$$

Also, the variations of the non-conservative work and the virtual momentum transport at the right boundary are

$$\delta W_{n.c.}=F_c(t)\,\delta w(L,\ t) \\ -d_a w_t(L,\ t)\,\delta w(L,\ t) \tag{8}$$

$$\delta W_{r.b.}=-\rho Av\{w_t(L,\ z) \\ +vw_x(L,\ t)\}\delta w(L,\ t) \tag{9}$$

where d_a is the damping coefficient of the actuator, and $F_c(t)$ is the control force.

The substitution of (6)-(9) into (2) yields the equations of motion in the following form.

$$\rho AW_{tt}+2\rho Avw_{st}-(T_0-\rho Av^2)\,w_{xx}=0 \tag{10}$$

where boundary conditions are

$$w(0,\ t)=0 \tag{11}$$

$$mw_{tt}(L,\ t)+d_a w_t(L,\ t) \\ +T_0 w_x(L,\ t)=F_c(t) \tag{12}$$

Note that in (10), the first, second, and third terms represent the local, Coriolis, and centripetal accelerations, respectively. Note also that (10) is a partial differential equation representing the transverse motion, whereas (12) is an ordinary differential equation representing the actuator dynamics at the right boundary that is coupled with the string tension and the control force.

Mote (1965) revealed that the string moving speed v, to avoid a divergence of the solution,

Table 1 The plant parameters used in simulations

Symbols	Definitions	Values
A	cross-sectional area	1.4×0.0045 [m^2]
L	length of the string	20 [m]
T_0	tension of the string	9,800 [kN]
m	mass of the actuator	25 [kg]
v	string moving speed	1.8 [m/s]
ρ	mass per unit volume	7,850 [kg/m^3]
d_a	damping coefficient	50 [Ns/m]

should be smaller than some critical speed given by

$$0<v<v_{cr}=\sqrt{\frac{T_0}{\rho A}} \tag{13}$$

Hence, the satisfaction of (13) is also assumed in this paper. If using the parameters in Table 1, $v_{cr}=\sqrt{T_0/\rho A}=445.15$ m/sec is given.

To actively control the transverse vibrations, a hydraulic touch roll is attached to the right end of the string. The two rollers of the touch roll can rotate freely, which allows the string to move freely in the axial direction without friction. But, the contact between the string and the rollers is tight enough, so that the displacement of the roller is considered as the displacement of the string. As seen in Fig. 1, the control input to the system is the current to the electro-hydraulic servo-valve. Hence, the dynamics of the hydraulic servo-valve as well as the dynamics of the touch roll together have to be included in the control system design (Alleyne and Liu, 2000 ; Araki and Taguchi, 2003 ; Goodwin and Quevedo, 2003).

Regarding the touch roll as a second order mass-damper system and including only the second order dynamics of the hydraulic system, the dynamics of the electro-hydraulic servo system is given by

$$\dot{x}_1=x_2 \tag{14}$$

$$\dot{x}_2=\frac{1}{m}(A_a x_3-d_a x_2-T_0 w_x) \tag{15}$$

$$\dot{x}_3=-\alpha x_2-\beta x_3+(\gamma\sqrt{P_s-\ \mathrm{sgn}(x_4)x_3}\,)x_4 \tag{16}$$

$$\dot{x}_4=\frac{1}{\tau}x_4+\frac{K}{\tau}u \tag{17}$$

where

$$\alpha=4A_a\beta_e/V_t,\ \beta=4C_{tm}\beta_e/V_t$$
$$\gamma=4C_d\beta_e w_g/(V_t\sqrt{\rho_f}) \tag{18}$$

$x_1=w(L,t)$ is the displacement of the actuator, $x_2=w_t(L,t)$ is the velocity of the actuator, $x_3=P_L$ is the load pressure, $x_4=x_v$ is the servo-valve position, u is the input current to the servo-valve, P_s is the supply pressure, β_e is the effective bulk modulus, V_t is the actuator total volume, C_{tm} is the coefficient of leakage, C_d is the discharge coefficient, w_g is the spool valve area gradient, ρ_f is the fluid density, τ is the time constant, K is a gain of the valve, and finally A_a is the cross-section area of the actuator.

3. Boundary Control Law

In this section, a right boundary control law that suppresses the transverse vibration of the string governed by (10)-(12) and (14)-(17) is derived. The following lemmas are first stated.

Lemma 1 : The mechanical energy of the string

$$V_{String}=\frac{\rho A}{2}\int_0^L (w_t+vw_x)^2dx+\frac{T_0}{2}\int_0^L w_x^2dx \tag{19}$$

and the functional $\tilde{V}$ defined by

$$\tilde{V}=V_{String}+V_B \tag{20}$$

are equivalent, where

$$V_B=\rho A\delta\int_0^L xw_x(w_t+vw_x)\,dx \tag{21}$$

That is, there exists a constant $\delta>0$ such that

$$(1-C_1)\,V_{String}\leq\tilde{V}\leq(1+C_1)\,V_{String} \tag{22}$$

where $0<C_1<1$.

Proof :

$$\begin{aligned} V_B&=\rho A\delta\int_0^L xw_x(w_t+vw_x)\,dx\\ &\leq\frac{\rho A\delta L}{2}\left\{\int_0^L w_x^2dX+\int_0^L(w_t+vw_x)^2dx\right\}\\ &=\delta L\left\{\frac{\delta A}{2}\int_0^L(w_t+vw_x)^2dx\right\}+\delta L\left(\frac{\rho A}{T_0}\frac{T_0}{2}\int_0^L w_x^2dx\right)\\ &\leq\delta L\cdot\max\left\{1,\frac{\rho A}{T_0}\right\}V_{String}=C_1V_{String}\end{aligned} \tag{23}$$

where

$$C_1=\delta L\cdot\max\left\{1, \frac{\rho A}{T_0}\right\}>0 \tag{24}$$

Hence, the following holds:

$$-C_1 V_{String}\le V_V\le C_1 V_{String} \tag{25}$$

By adding V_{String} to the both sides of (25), (25) becomes

$$(1-C_1)\ ]\ V_{String}\le \tilde{V}\le(1+C_1)\,V_{String} \tag{26}$$

In order for V_{String} and $\tilde{V}$ to be equivalent, $1-C_1>0$ is needed. From (24),

$$0<\delta<\frac{1}{L\cdot\max\left\{1, \frac{\rho A}{T_0}\right\}} \tag{27}$$

needs to be satisfied. Lemma 1 is proved. ■

Let the errors of the load pressure and the servo-valve position from their desired values, respectively, be defined by

$$e_3=x_3-x_{3desired} \tag{28}$$

$$e_4=x_4-x_{4desired} \tag{29}$$

Now, with Lemma 1, the following Lyapunov function candidate, which is basically equivalent to the total mechanical energy of the string and actuator, is proposed as

$$V(t)=\tilde{V}+V_{Actuator} \tag{30}$$

where

$$V_{Actuator}=\frac{m}{2}\{w_t(L,t)+\psi(v+\delta L)\,w_x(L,t)\}^2 + \frac{1}{2}e_3^2+\frac{1}{2}e_e^2,\ \psi>0 \tag{31}$$

It is noted that because the considered system involves a mass flow entering in and out at the boundaries, the net change of the total energy is the sum of the change in the control volume (i.e., $\frac{\partial}{\partial t}V_{String}$) and the energy flux at the boundaries (i.e., $v\,V_{String}|_0^L$). The time derivative of (30) can be derived by applying the Reynolds transport theorem as

$$\frac{d}{dt}V(t)=V_t+v\,V_x|_0^L \tag{32}$$

First, the total derivative (or the material derivative) of (30) is evaluated. The time derivative of the first term in (30), $\tilde{V}$, becomes

$$\frac{d}{dt}\tilde{V}(t)=\frac{d}{dt}V_{String}+\frac{d}{dt}V_B \tag{33}$$

By using (10), $\frac{d}{dt}V_{String}(t)=V_{(String)_t}+v\,V_{(String)_x}$ can be evaluated

$$\begin{aligned} V_{(String)_t}&=\int_0^L \rho A(w_t+vw_x)(w_{tt}+vw_{xt})\,dx \\ &\quad+\int_0^L T_0 w_x w_{xt}dx \\ &=\int_0^L (w_t+vw_x)\{(T_0-\rho Av^2)\,w_{xx}\}dx \\ &\quad-\int_0^L \rho Avw_{xt}(w_t+vw_x)\,dx \\ &\quad+\int_0^L T_0 w_x w_{xt}dx \\ &=(T_0-\rho Av^2)\,[w_x w_t]_0^L \\ &\quad+\frac{v(T_0-\rho Av^2)}{2}[w_x^2]-\frac{\rho Av}{2}[w_t^2]_0^L \end{aligned} \tag{34a}$$

$$\begin{aligned} v\,V_{(String)_x}&=v\int_0^L \rho A(w_t+vw_x)(w_{xt}+vw_{xx})\,dx \\ &\quad+v\int_0^L T_0 w_x w_{xx}dx \\ &=\frac{\rho Av}{2}[(w_t+vw_x)^2]_0^L+\frac{vT_0}{2}[w_x^2]_0^L \end{aligned} \tag{34b}$$

Hence,

$$\begin{aligned} \frac{d}{dt}V_{String}&=(T_0-\rho Av^2)\,[w_x w_t]_0^L+\frac{v(T_0-\rho Av^2)}{2}[w_x^2]_0^L \\ &\quad-\frac{\rho Av}{2}[w_t^2]_0^L+\frac{\rho Av}{2}[(w_t+vw_x)^2]_0^L \\ &\quad+\frac{vT_0}{2}[w_x^2]_0^L \\ &=T_0 w_x(L,t)\,w_t(L,t)+vT_0[w_x^2]_0^L \end{aligned} \tag{35}$$

Also, for the second term in (33), the following hold:

$$\begin{aligned} V_{(B)_t}&=\rho A\delta\int_0^L xw_{xt}(w_t+vw_x)\,dx \\ &\quad+\rho A\delta\int_0^L xw_x(w_{tt}+vw_{xt})\,dx \end{aligned} \tag{36a}$$

$$\begin{aligned} v\,V_{(B)_x}&=v\rho A\delta\int_0^L w_x(w_t+vw_x)\,dx \\ &\quad+v\rho A\delta\int_0^L xw_{xx}(w_t+vw_x)\,dx \\ &\quad+v\rho A\delta\int_0^L xw_x(w_{xt}+vw_{xx})\,dx \end{aligned} \tag{36b}$$

Therefore,

$$\frac{d}{dt}V_B=v\rho A\delta\int_0^L(xw_xw_{xt}+xw_tw_{xx}+w_xw_t)\,dx$$
$$+v^2\rho A\delta\int_0^L xw_xw_{xx}dx+\rho A\delta\int_0^L xw_{xt}w_tdx$$
$$+\delta\int_0^L xw_x(\rho Aw_{tt}+2\rho Avw_xt+\rho Av^2w_{xx})\,dx \quad (37)$$
$$+v^2\rho A\delta\int_0^L w_x^2dX$$

Lemma 2 : Because $w(x,\ t)$ should satisfy the boundary condition (11), the following relations hold :

$$\int_0^L xw_{xt}w_tdx=\frac{L}{2}\,w_i^2(L,\ t)-\frac{1}{2}\int_0^L w_t^2dx \quad (38a)$$

$$\int_0^L xw_xw_{xx}dx=\frac{L}{2}\,w_x^2(L,\ t)-\frac{1}{2}\int_0^L w_xdx \quad (38b)$$

$$\int_0^L(xw_{xt}w_x+xw_{xx}w_t+w_xw_t)\,dx$$
$$=[xw_xw_t]_0^L=Lw_x(L,\ t)\,w_t(L,\ t) \quad (38c)$$

Proof : The integration by parts gives the above equalities immediately. ■

Now, by using Lemma 2, (33) is modified as

$$\frac{d}{dt}\tilde{V}(t)=T_0w_x(L,\ t)\,w_t(L,\ t)+vT_0[w_x^2]_0^L$$
$$+\rho Av\delta Lw_x(L,\ t)\,w_t(L,\ t)$$
$$+\frac{1}{2}(\rho Av^2\delta L+T_0\delta L)\,w_x^2(L,\ t) \quad (39)$$
$$+\frac{\rho Av^2\delta}{2}\int_0^L w_x^2dx+\frac{\rho A\delta L}{2}\,w_t^2(L,\ t)$$
$$-\frac{\rho A\delta}{2}\int_0^L w_t^2dx-\frac{\delta T_0}{2}\int_0^L w_x^2dx$$

Also, the time derivative of (31) becomes

$$\frac{d}{dt}V_{Actuator}=\{w_t(L,\ t)+\psi(v+\delta L)\,w_x(L,\ t)\}$$
$$\{mw_{tt}(L,\ t)+\psi m(v+\delta L)\,w_{xt}(L,\ t)\} \quad (40)$$
$$+e_3\dot{e}_3+e_4\dot{e}_4$$

Let the right boundary control force $F_c(t)$ be

$$F_c(t)=-\psi m(v+\delta L)\,w_{xt}(L,\ t) \quad (41)$$

where ψ is the control gain. Using (12), (40) can be rewritten as

$$\frac{d}{dt}V_{Actuator}=\{w_t(L,\ t)+\psi(v+\delta L)\,w_x(L,\ t)\}$$
$$\{-d_aw_t(L,\ t)-T_0w_x(L,\ t)\}+e_3\dot{e}_3+e_4\dot{e}_4$$
$$=\{w_t(L,\ t)+\psi(v+\delta L)\,w_x(L,\ t)\}$$
$$\{-d_aw_t(L,\ t)-T_0w_x(L,\ t)\} \quad (42)$$
$$+e_3(\alpha x_2+\beta x_3+\gamma\sqrt{P_S-\mathrm{sgn}(x_4)\,x_3}\,x_{4desired}$$
$$+\gamma\sqrt{P_S-\mathrm{sgn}(x_4)\,x_3}\,e_4-\dot{x}_{3desired})$$
$$+e_4\left(-\frac{x^4}{\tau}+\frac{K}{\tau}\,u-\dot{x}_{4desired}\right)$$

The desired position of the servo-valve, $X_{desired}$, is defined as

$$x_{4desired}=\frac{1}{\gamma\sqrt{P_S-\mathrm{sgn}(x_4)\,x_4}} \quad (43)$$
$$\{-(\alpha x_2-\beta x_3)+\dot{x}_{3desired}-s_3e_3\}$$

where $s_3>0$. The substitution of (43) into (42) yields :

$$\frac{d}{dt}V_{Actuator}=\{w_t(L,\ t)+\psi(v+\delta L)\,w_x(L,\ t)\}$$
$$\{-d_aw_t(L,\ t)-T_0w_x(L,\ t)\} \quad (44)$$
$$-s_3e_3^2+\gamma\sqrt{P_S-\mathrm{sgn}(x_4)\,x_3}\,e_3e_4$$
$$+e_4\left(-\frac{x_4}{\tau}+\frac{K}{\tau}\,u-\dot{x}_{4desired}\right)$$

Finally, from (39) and (44), the total derivative of (30) becomes

$$\frac{d}{dt}V(t)=\frac{d}{dt}(\tilde{V}+V_{Actuator})\le T_0w_x(L,\ t)\,w_t(L,\ t)$$
$$+vT_0[w_x^2]_0^L+\rho Av\delta Lw_x(L,\ t)\,w_t(L,\ t)$$
$$+\frac{1}{2}(\rho Av^2\delta L+T_0\delta L)\,w_x^2(L,\ t)$$
$$+\frac{\rho Av^2\delta}{2}\int_0^L w_x^2dx+\frac{\rho A\delta L}{2}\,\delta_t^2(L,\ t)$$
$$-\frac{\rho A\delta}{2}\int_0^L w_t^2dx-\frac{\delta T_0}{2}\int_0^L w_x^2dx-d_aw_t^2(L,\ t) \quad (45)$$
$$-d_a\psi(v+\delta L)\,w_x(L,\ t)\,w_t(L,\ t)$$
$$-T_0w_x(L,\ t)\,w_t(L,\ t)-T_0\psi(v+\delta L)\,w_x^2(L,\ t)$$
$$-s_3e_3^2+\gamma\sqrt{P_S-\mathrm{sgn}(x_4)\,x_3}\,e_3e_4$$
$$+e_4\left(-\frac{x_4}{\tau}+\frac{K}{\tau}\,u-\dot{x}_{4desired}\right)$$

Now, the main theorem of this paper is stated as follows :

Theorem : Consider the system (10)-(12) and (14) (17). Let the right boundary control force $F_c(t)$ be given by $F_c(t)=-\psi m(v+\delta L)\,w_{xt}(L,\ t)$, where ψ is the control gain. Let the damping coefficient of the actuator d_a in (12) be given by

$$d_a \geq \max\left\{\frac{\rho A \delta L}{2}, \frac{\rho A \delta L}{\psi(1+\delta L/v)}\right\} \quad (46)$$

Assume further that the following conditions are satisfied :

$$1-\psi<0, \ \rho A v^2 - T_0 < 0$$

and

$$\frac{1}{2}\rho A v^2 + T_0 - T_0\psi < 0 \quad (47)$$

Then, the closed loop system with the following control input is exponentially stable :

$$u=\frac{\tau}{K}\left[-\left(-\frac{x_4}{\tau}\right)+\dot{x}_{4desired}-s_4e_4-\gamma\sqrt{P_S-\mathrm{sgn}(x_4)x_3}e_3\right]$$
$$=\frac{\tau}{K}\left[\frac{x_v}{\tau}+\dot{x}_{vdesired}-s_4(x_v-x_{vdesired}) -\gamma\sqrt{P_S-\mathrm{sgn}(x_v)P_L}(P_L-P_{Ldesired})\right] \quad (48)$$

where τ and K are the time constant and the gain of the servo-valve, respectively,

$$P_{Ldesired}=\frac{\psi m(v+\delta L)w_{xt}}{A_a} \quad (49)$$

$$x_{vdesired}=\frac{1}{\gamma\sqrt{P_S-\mathrm{sgn}(x_v)P_L}} \times\{\alpha w_t(L,t)+\beta P_L+\dot{P}_{Ldesired}-s_3(P_L-P_{Ldesired})\} \quad (50)$$

$$\dot{P}_{Ldesired}=-\frac{\psi m(v+\delta L)w_{xtt}}{A_a} \quad (51)$$

$\alpha=4A_a\beta_e/V_t$, $\beta=4C_{tm}\beta_e/V_t$, $\gamma=4C_d\beta_e w_g/(V_t\sqrt{\rho_f})$, and $s_3, s_4>0$.

Proof : The proof is given in Section 4. ■

Remark : The satisfaction of (46) and (47) is immediately seen : Specifically, if using the parameter values in Table 1, δ is first calculated from (27) as follows :

$$\delta<\frac{1}{20\cdot\max\left\{1, \frac{7{,}850\times1.4\times0.0045}{9{,}800{,}000}\right\}}=0.05$$

So, let $\delta=0.04$, see (Rao, 1990). Then, from (46), d_a is calculated as follows :

$$d_a\geq\max\left\{\frac{7.850\times1.4\times0.0045\times0.04\times20}{2}, \frac{7.850\times1.4\times0.0045\times0.04\times20}{100(1+0.04\times20/1.8)}\right\}$$
$$\geq\max\{19.782, 0.2739\}$$

where the control gain $\psi=100$ has been assumed. Also, for (47)

$$\rho A v^2 - T_0 = 7.850\times1.4\times0.0045\times1.8^2-9{,}800 = -9{,}639.7<0$$

$$\frac{1}{2}\rho A v^2+T_0-T_0\psi$$
$$=\frac{1}{2}\times7{,}850\times1.4\times0.0045\times1.8^2 +9{,}800{,}000-9{,}800{,}000\times100$$
$$=-970{,}199{,}919.9<0$$

Hence, all the conditions in the Theorem are well satisfied.

4. Stability Analysis

In this section, the exponential stability of the string system under control input (41) and damping coefficient (46) is proved. Note that the time derivative of the Lyapunov function (i.e., (45) under the conditions of the Theorem) is expressed by

$$\frac{d}{dt}V(t)\leq X+Y \quad (52)$$

where

$$X=-\left(\frac{\rho A v\delta L}{\psi(v+\delta_L)}-\frac{\rho A\delta L}{2}\right)w_t^2(L,t) -\frac{\delta L}{2}(T_0-\rho A v^2)w_x^2(L,t)-s_3e_3^2-\frac{1}{\tau}e_4^2 \quad (53)$$

$$Y=-\frac{\rho A\delta}{2}\int_0^L w_t^2 d\mathrm{X}-\frac{\delta T_0}{2}\int_0^L w_x^2 dx \quad (54)$$

At first, X is

$$X=-\left(\frac{\rho Av\delta L}{\psi(v+\delta L)}-\frac{\rho A\delta\delta L}{2}\right)w_t^2(L,t)-\frac{\delta L}{2}(T_0-\rho Av^2)w_x^2(L,t)-s_3e_3^2-\frac{1}{\tau}e_4^2$$
$$=-\frac{1}{m}\left(\frac{\rho Av\delta L}{\psi(v+\delta L)}-\frac{\rho A\delta L}{2}\right)mw_t^2(L,t)$$
$$-\frac{L\delta}{2\psi m(v+\delta L)^2}(T_0-\rho Av^2)\psi m(v+\delta L)^2w_x^2(L,t)-2s_3\cdot\frac{1}{2}e_3^2-\frac{2}{\tau}\cdot\frac{1}{2}e_4^2 \quad (55)$$
$$\leq-\min(C_2,C_3,C_4,C_5)\left\{mw_t^2(L,t)+\psi m(v+\delta L)^2w_x^2(L,t)+\frac{1}{2}e_3^2+\frac{1}{2}e_4^2\right\}$$
$$\leq\min(C_2,C_2,C_4,C_5)\left[\frac{m}{2}\{w_t(L,t)+\psi(v+\delta L)w_x(L,t)\}^2+\frac{1}{2}e_3^2+\frac{1}{2}e_4^2\right]$$
$$=-\min(C_2,C_3,C_4,C_o)V_{Actuator}$$

where,

$$C_2=\frac{1}{m}\left(\frac{\rho Av\delta L}{\psi(v+\delta L)}-\frac{\rho A\delta L}{2}\right)$$

$$C_3=\frac{\delta L}{2\psi m(v+\delta L)^2}(T_0-\rho Av^2)$$

$$C_4=2s_3 \text{ and } C_5=\frac{2}{\tau}$$

And, Y is

$$\begin{aligned}
Y&=-\frac{\rho A\delta}{2}\int_0^L w_t^2dx-\frac{\delta T_0}{2}\int_0^L w_x^2dx\\
&=-\frac{\delta T_0}{4}\int_0^L w_x^2dx-\frac{\rho A\delta}{2}\int_0^L w_t^2dx-\frac{\delta T_0}{4}\int_0^L w_x^2dx\\
&\le-\frac{\delta T_0}{4}\int_0^L w_x^2dx-\rho A\delta\cdot\frac{1}{2}\int_0^L w_t^2dx-\frac{\delta T_0}{2v^2}\cdot\frac{v^2}{2}\int_0^L w_x^2dx\\
&\le-\frac{\delta T_0}{4}\int_0^L w_x^2dx-\min(C_6, C_7)\left(\int_0^L w_t^2dx+v^2\int_0^L w_x^2dx\right)\\
&\le-\frac{\delta T_0}{4}\int_0^L w_x^2dx-\min(C_6, C_h)\cdot\frac{1}{2}\int_0^L (w_t+vw_x)^2dx\\
&\le-\min(C_6, C_7, C_8)\left(\frac{T_0}{2}\int_0^L w_x^2dx+\frac{1}{2}\int_0^L (w_t+vw_x)^2dx\right)\\
&=-\min(C_6, C_7, C_\pi)\,V_{String}
\end{aligned}\tag{56}$$

where $C_6=\frac{\rho A\delta}{2}$, $C_7=\frac{\delta T_0}{4v^2}$, $C_\pi=\frac{\delta}{2}$. By using (20), (56) is expressed as

$$Y\le-\frac{\min(C_6, C_7, C_8)}{1+C_1}(V_{String}+V_B)\tag{57}$$

Then, from (55) and (57), the time derivative of the Lyapunov function (52) can be expressed as

$$\begin{aligned}
\frac{d}{dt}V(t)&\le X+Y\le-\min(C_2, C_3, C_4, C_5)\,V_{Actuator}\\
&\quad-\frac{\min(C_6, C_7, C_\pi)}{1+C_1}(V_{String}+V_V)\\
&=-\lambda V(t)
\end{aligned}\tag{58}$$

where

$$\lambda=\min\left(C_2, C_3, C_4, C_5, \frac{C_6}{1+C_1}, \frac{C_7}{1+C_1}, \frac{C_8}{1+C_1}\right)$$

Hence, the Lyapunov functional is decaying exponentially in time. This implies that the total mechanical energy (19) of the string decays exponentially in time, which again implies that all the state variables decay exponentially in time. Hence, the closed loop system is exponentially stable.

5. Implementation and Simulations

The implementation of (41) and (46) requires two things : the generation of control force $F_c(t)$ and the satisfaction of a damping coefficient d_a. Because the satisfaction of d_a is related to the design problem of a hydraulic actuator, it must be pre-planned. Note that δ should satisfy (27). Because all terms in the right- and left-hand sides of (27) are already known, the range of the damping coefficient can be achieved. The implementation of $w_{xt}(L, t)$ in (41) can be achieved by backwards differencing of $w_x(L, t)$ measured at each step.

To demonstrate the performance of the closed loop system, computer simulations using the finite difference scheme have been performed. The plant parameters used for simulations are gathered in Table 1. And, the used servo-valve parameters are collected in Table 2. The higher d_a is, the faster the exponential decay is. The use of $d_a=50$ Ns/m is recommended.

Let the initial conditions be

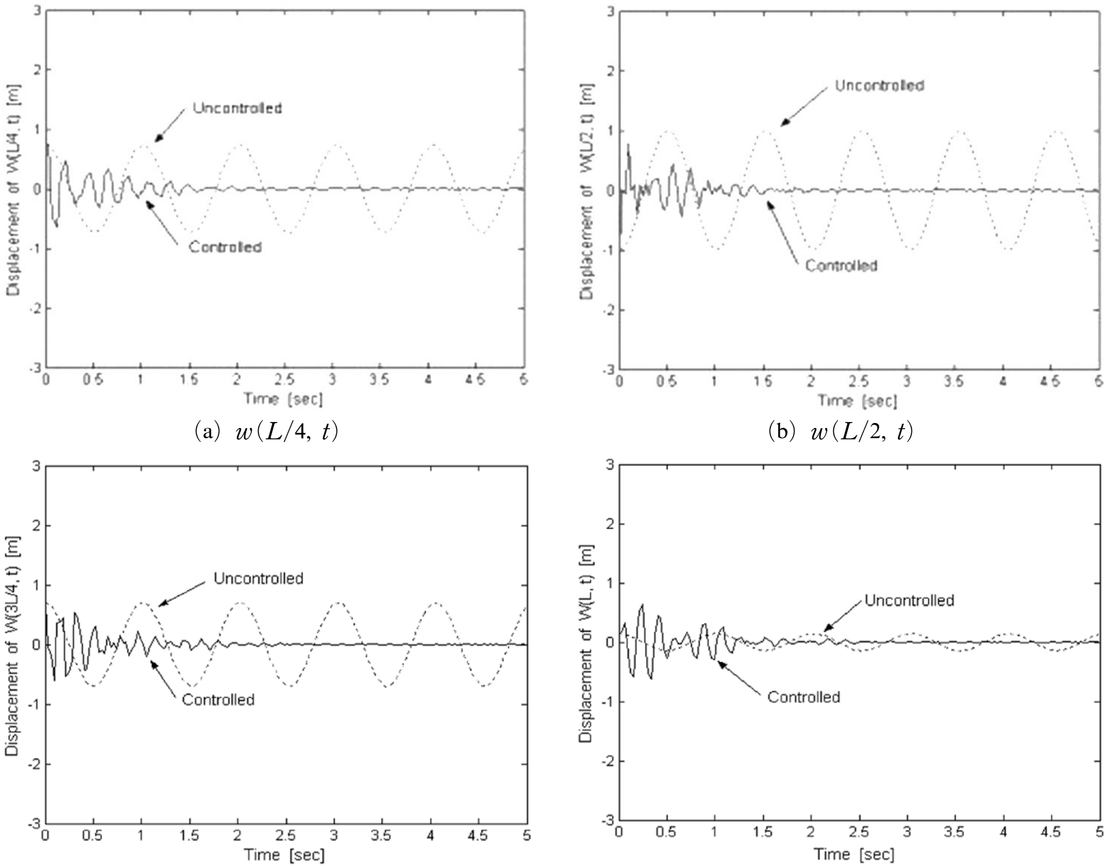
$$w(x, 0)=\sin(3\pi x) \text{ and } w_t(x, 0)=0\tag{59}$$

Now, simulations using $\delta=0.04$, $d_a=50$, $\psi=100$, and (59) have been performed for 5 seconds. Figure 2 shows the transverse displacement at $x=L/4$, $L/2$, $3L/4$, and $L=20$ m, respectively. Figure 3 shows the applied control force and its desired value at $x=L$, respectively. As shown in Fig. 2, the lateral vibration has been suppressed

Table 2 The servo-valve and design parameters used in simulations

Symbols	Definitions	Values
P_s	load pressure	1.0344×10^7 [Pa]
A_a	actuator ram area	3.2673×10^{-4} [m^2]
α	$4A_a\beta_e/V_t$	1.513×10^{10} [N/m^3]
β	$4C_{im}\beta_e/V$	1.0 [1/s]
γ	$4C_d\beta_e w_g/(V_t\sqrt{\rho_f})$	8.0×10^8
s_3	positive constant	2,000
s_4	positive constant	500

within 3 seconds. Figure 4 shows the exponential decay of the total mechanical energy of the closed- loop system, whereas the energy without control remains at the same level in time.



(a) $w(L/4, t)$ (b) $w(L/2, t)$

(c) $w(3L/4, t)$ (d) $w(L, t)$

Fig. 2 The transverse displacements with damping coefficient $d_a=50$: (a) $w(L/4, t)$ (b) $w(L/2, t)$ (c) $w(3L/4, t)$, and (d) $w(L, t)$, where $L=20$ m

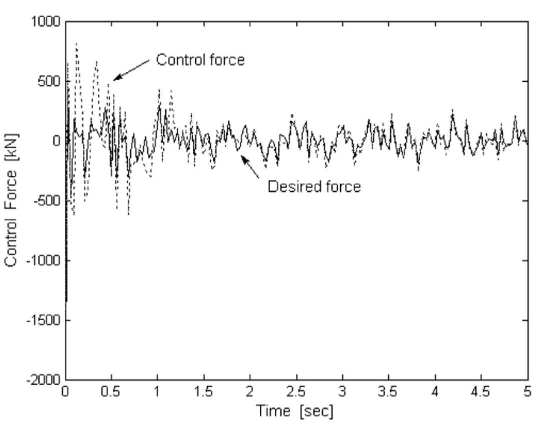


Fig. 3 The desired and applied control forces in Fig. 2

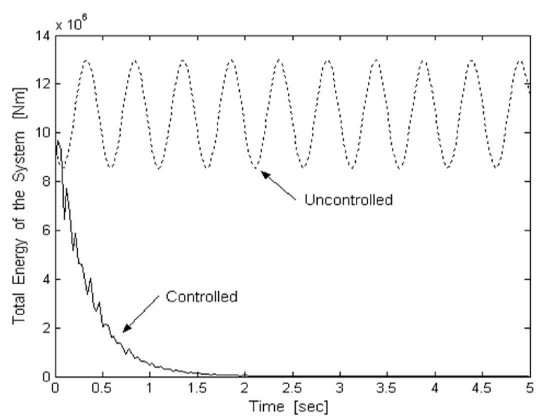


Fig. 4 The exponential decay of the total energy

6. Conclusions

This paper investigated a boundary control law for suppressing the transverse vibration of an axially moving string. Because the control law was derived for a general axially moving linear string equation, it can be applied to various engineering problems, for instance, a moving steel strip in the zinc galvanizing line, a thin metal-wire production line, etc. Achieving the exponential stability of the closed-loop system by using one sensor and one actuator is the main contribution of the proposed algorithm.

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